

UPPER SEMI-CONTINUITY OF THE HILBERT-KUNZ MULTIPLICITY

ILYA SMIRNOV

ABSTRACT. We prove that the Hilbert-Kunz multiplicity is upper semi-continuous in F-finite rings and algebras of essentially finite type over an excellent local ring.

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1. INTRODUCTION

Let R be a commutative Noetherian ring of characteristic $p > 0$. For a prime ideal \mathfrak{p} , the (normalized) Hilbert-Kunz function of \mathfrak{p} is defined to be

$$e \mapsto \frac{\ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}^{[p^e]}R_{\mathfrak{p}})}{p^{ed}},$$

where e is a positive integer and $\mathfrak{p}^{[p^e]}$ is the ideal generated by the p^e th powers of elements of \mathfrak{p} . The Hilbert-Kunz multiplicity of \mathfrak{p} is defined as the limit of this sequence. Hilbert-Kunz theory originates in the work of Kunz ([9, 10]) and Hilbert-Kunz multiplicity was introduced by Monsky ([12]) in 1983.

From the beginning, there was a perception that Hilbert-Kunz theory should be a measure of singularities. In fact, in 1969 Kunz has shown that Hilbert-Kunz function characterizes singularity, and then, in 2000, Watanabe and Yoshida generalized this for Hilbert-Kunz multiplicity. In [16] they showed that an unmixed local ring (R, \mathfrak{m}) is regular if and only if $e_{HK}(\mathfrak{m}) = 1$.

Since it follows from the work of Kunz that $e_{HK}(R) \geq 1$, one may expect that when $e_{HK}(R)$ is getting close to R , the singularity of R is getting better. A notable example of this was given by Blickle and Enescu in [2] and then improved by Aberbach and Enescu in

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[1]. They show that if $e_{HK}(R)$ is sufficiently close to 1, then R has to be Gorenstein and F-regular.

This work is devoted to a global property of Hilbert-Kunz multiplicity: upper semi-continuity. A formal definition of upper semi-continuity is given in Definition 1, but via Nagata's criterion (Proposition 8) this question can be considered as a distribution property of singularities of local rings of R . Namely, given a prime ideal \mathfrak{p} of R , we want to know if the Hilbert-Kunz multiplicity of a generic prime containing \mathfrak{p} is close to the Hilbert-Kunz multiplicity of $R_{\mathfrak{p}}$, see Proposition 9. So, we may say, that we are trying to understand if Hilbert-Kunz multiplicity sees that the singularity of a general prime containing \mathfrak{p} is close to that of \mathfrak{p} .

We also want to bring reader's attention to Example 7, which shows that the Hilbert-Kunz multiplicity of a general prime containing \mathfrak{p} need not to be equal to $e_{HK}(\mathfrak{p})$. This illustrates the subtlety of the problem. Also, note that the corresponding statement for the Hilbert-Samuel multiplicity is true.

In 1976 ([10]) Kunz proved that for a fixed e the Hilbert-Kunz function $\ell(R_{\mathfrak{p}}/\mathfrak{p}^{[p^e]}R_{\mathfrak{p}})/p^{ed}$ is upper semi-continuous, but his methods do not extend for the limit. In [4] Enescu and Shimomoto asked whether Hilbert-Kunz multiplicity is upper semi-continuous. Also they proved that Hilbert-Kunz multiplicity is dense upper semi-continuous on the maximum spectrum, which is a weaker condition. Later, in 2011, there was a group working on this question in the AIM workshop "Relating test ideals and multiplier ideals".

We prove that Hilbert-Kunz multiplicity is upper semi-continuous in locally equidimensional F-finite rings and locally equidimensional rings of essentially finite type over an excellent local ring, a mild restriction that is satisfied by complete local domains and domains finitely generated over a field. To achieve this, we want to control the convergence rate of the Hilbert-Kunz function and, building on Tucker's estimates from [15], we show that it can be controlled generically in Theorem 18 for F-finite domains and in Theorem 21 for algebras of essentially finite type over a complete local domain. This method allows us to reduce upper semi-continuity of the limit of a sequence (Hilbert-Kunz multiplicity) to upper semi-continuity of a term of the sequence (a fixed Hilbert-Kunz function), and the latter is known by the work of Kunz. This strategy should be useful for other numerical invariants in positive characteristic, for example for F-signature. However, an application of this approach seems to require a better understanding of F-signature, see Remark 25.

The structure of the proof is as following. We start with general preliminaries in Section 2. Then we develop the machinery of global convergence estimates in Sections 3 and 4. Section 3 treats F-finite case and Section 4 obtains the same result for algebras of essentially finite type over a complete local domain, so a reader interested only in F-finite case may skip it. Then the estimates are used in Section 5 to prove the main theorem.

2. PRELIMINARIES

In this paper all rings assumed to be commutative Noetherian and containing an identity element. For a module M over a ring R , we will use $\ell(M)$ to denote the length of M .

Let R be a ring of characteristic $p > 0$. For convenience, we use $q = p^e$ where e may vary. For an ideal I of R , let $I^{[q]}$ be the ideal generated by q th powers of the elements of I . By

F_*R we mean R viewed as an R -module via the extension of scalars through the Frobenius endomorphism. If R is reduced, F_*R can be identified with the ring of p -roots $R^{1/p}$. We say that R is F-finite if F_*R is a finitely generated R -module.

Definition 1. Let X be a topological space. A real-valued function f is upper semi-continuous if for any $a \in \mathbb{R}$ the set $\{x \in X \mid f(x) < a\}$ is open in X .

In his papers [9] and [10], Kunz initiated the study of the Hilbert-Kunz function $f_q(R) = \frac{1}{q^d} \ell(R/\mathfrak{m}^{[q]})$. For any q , we can define a function on $\text{Spec } R$, the spectrum of R , by setting $f_q(\mathfrak{p}) = f_q(R_{\mathfrak{p}})$. In [10, Proposition 3.3, Corollary 3.4] Kunz obtained the following results.

Theorem 2. *If R is a locally equidimensional ring, then for all q*

- (1) $f_q(\mathfrak{p}) \leq f_q(\mathfrak{q})$, if $\mathfrak{p} \subseteq \mathfrak{q}$,
- (2) $f_q(\mathfrak{p})$ is upper semi-continuous on $\text{Spec } R$.

We should note, that Kunz claimed this result for an equidimensional ring, but Shepherd-Barron pointed out in [14] that the theorem is false if R is not locally equidimensional.

Definition 3. Let (R, \mathfrak{m}) be a local ring of characteristic $p > 0$. It was shown by Monsky that the limit

$$e_{HK}(R) = \lim_{q \rightarrow \infty} \frac{\ell(R/\mathfrak{m}^{[q]})}{q^{\dim R}}$$

exists and is called the Hilbert-Kunz multiplicity of R .

Defining $e_{HK}(\mathfrak{p}) := e_{HK}(R_{\mathfrak{p}})$, we can view the Hilbert-Kunz multiplicity as a function on $\text{Spec } R$. In view of Kunz's result, it was natural to pose the following conjecture.

Conjecture 4. *If R is a locally equidimensional excellent ring, then the Hilbert-Kunz multiplicity is an upper semi-continuous function on $\text{Spec } R$.*

Or, less generally, if R be a locally equidimensional F-finite ring, then the Hilbert-Kunz multiplicity is an upper semi-continuous function on $\text{Spec } R$.

Note that F-finite rings are excellent by a theorem of Kunz ([10][Theorem 2.5]).

Also, we want to note that the restriction of the second statement in the conjecture is somehow natural, since F-finite rings are much easier to work with. For example, it is still not known whether all excellent rings have a test element.

Remark 5. The reader should be warned that Shepherd-Barron (in [14][Corollary 2]) claimed a much stronger statement. However, in his proof, he used that a descending sequence of open sets stabilizes without a proper justification. In fact, Shepherd-Barron's claim implies that e_{HK} attains only finitely many values on $\text{Spec } R$. But the author was able to give a counter-example to this claim, see Example 7.

It is also worth pointing out that a stronger property holds for Hilbert-Kunz functions. The proofs of semi-continuity by Kunz ([10]) and Shepherd-Barron ([14]) show the following statement.

Proposition 6. *Let R be a locally equidimensional ring and \mathfrak{p} be a prime ideal. Then for any fixed q and any $a \in \mathbb{R}$ the set*

$$\{\mathfrak{p} \mid f_q(\mathfrak{p}) \leq a\}$$

is open in $\text{Spec } R$.

Example 7. Let $R = F[x, y, z, t]/(z^4 + xyz^2 + (x^3 + y^3)z + tx^2y^2)$, where F is the algebraic closure of $\mathbb{Z}/2\mathbb{Z}$. In [3], Brenner and Monsky showed that tight closure does not commute with localization in this ring.

Let $\mathfrak{p} = (x, y, z)$, this is a prime ideal of dimension one in R . Building on the work of Monsky ([13]), the author was able to show that the Hilbert-Kunz multiplicity attains infinitely many values on the set of prime ideals containing \mathfrak{p} . Moreover, for any prime ideal \mathfrak{m} containing \mathfrak{p} , $e_{HK}(\mathfrak{m}) > e_{HK}(\mathfrak{p})$, so the set $\{\mathfrak{q} \mid e_{HK}(\mathfrak{q}) \leq e_{HK}(\mathfrak{p})\}$ is not open. The details will appear in a future paper.

This rather surprising result shows that, compared to the Hilbert-Samuel multiplicity and a fixed Hilbert-Kunz function, the Hilbert-Kunz multiplicity has a distinctively different global behavior.

We use the following standard terminology: a closed set $V(I)$ consists of all prime ideals containing $I \subseteq R$, a principal open set D_s consists of all prime ideals not containing $s \in R$. Since D_s can be naturally identified with $\text{Spec } R_s$, we will sometimes abuse notation, and, by saying to invert an element s , we will mean to consider D_s .

We recall Nagata's criterion of openness in $\text{Spec } R$ ([11, 22.B]).

Proposition 8. *Let R be a ring. A subset U of $\text{Spec } R$ is open if and only if*

- (1) *U is stable under generalization, i.e. if $\mathfrak{q} \in U$ and $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathfrak{p} \in U$,*
- (2) *U contains a nonempty open subset of $V(\mathfrak{p})$ for any $\mathfrak{p} \in U$.*

Since any open set is a union of principal open sets, the second condition is equivalent to $U \supseteq V(\mathfrak{p}) \cap D_s \neq \emptyset$ for some s .

From Theorem 2 it follows that if $\mathfrak{p} \subseteq \mathfrak{q}$ then $e_{HK}(\mathfrak{p}) \leq e_{HK}(\mathfrak{q})$, so for any a the set $\{\mathfrak{p} \in \text{Spec } R \mid e_{HK}(\mathfrak{p}) < a\}$ is stable under generalization. Hence, it is enough to verify only the second condition of the criterion. Thus we can restate Conjecture 4 in the following form.

Proposition 9. *Let R be a locally equidimensional ring. Then the Hilbert-Kunz multiplicity is upper semi-continuous on $\text{Spec } R$ if and only if for any prime ideal \mathfrak{p} and any $\varepsilon > 0$ there exists $s \notin \mathfrak{p}$ such that for all prime ideals $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$*

$$e_{HK}(\mathfrak{q}) < e_{HK}(\mathfrak{p}) + \varepsilon.$$

It is easy to show that we can restrict ourselves to domains.

Proposition 10. *Let R be a locally equidimensional ring. If the Hilbert-Kunz multiplicity is upper semi-continuous in R/\mathfrak{p} for all minimal primes \mathfrak{p} of R , then the Hilbert-Kunz multiplicity is upper semi-continuous in R .*

Proof. Given ε , we want to find an element $s \notin \mathfrak{p}$, such that for any ideal \mathfrak{q} containing $\mathfrak{p}R_s$ of R_s , $e_{HK}(\mathfrak{q}) < e_{HK}(\mathfrak{p}) + \varepsilon$.

For $i = 1 \dots n$ let \mathfrak{p}_i be the minimal primes of R . Inverting an element, we may assume that all \mathfrak{p}_i are contained in \mathfrak{p} . By the assumption, there exist elements $s_i \notin \mathfrak{p}$, such that in the corresponding subsets of $\text{Spec } R/\mathfrak{p}_i$,

$$e_{HK}(\mathfrak{q}R/\mathfrak{p}_i) < e_{HK}(\mathfrak{p}R/\mathfrak{p}_i) + \varepsilon/(n\ell_{R/\mathfrak{p}_i}(R/\mathfrak{p}_i)).$$

Now, if we invert the product s of s_i , we obtain that for any ideal \mathfrak{q} of R_s that contains \mathfrak{p} , by the associativity formula for Hilbert-Kunz multiplicity,

$$\begin{aligned} e_{HK}(\mathfrak{q}) &= \sum_{i=1}^n e_{HK}(\mathfrak{q}R/\mathfrak{p}_i) \ell_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}) < \\ &< \sum_{i=1}^n \left(e_{HK}(\mathfrak{p}R/\mathfrak{p}_i) + \frac{\varepsilon}{n \ell_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i})} \right) \ell_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}) = e_{HK}(\mathfrak{p}) + \varepsilon. \end{aligned}$$

□

Corollary 11. *Conjecture 4 holds if and only if for any excellent domain R , prime ideal \mathfrak{p} of R , and $\varepsilon > 0$, there exists $s \notin \mathfrak{p}$ such that for all prime ideals $\mathfrak{q} \in V(\mathfrak{p}) \cap D_s$,*

$$e_{HK}(\mathfrak{q}) < e_{HK}(\mathfrak{p}) + \varepsilon.$$

Proof. We just note that a quotient of an excellent ring is excellent. □

A descent of semi-continuity over a faithfully flat extension would be extremelly useful for the proof; in fact, it would eliminate the need of Section 4. Unfortunately, there is no good relation between the Hilbert-Kunz multiplicity of a local ring and its arbitrary faithfully flat extension. So, we are able to obtain a descent statement only for extensions with regular fibers. Still, the following lemma will be needed in the proof of our main result for algebras of essentially finite type over an excellent local ring.

Lemma 12. *Let R be a ring and $f: R \rightarrow S$ be a faithfully flat R -algebra. Moreover, suppose f has regular fibers. Then Hilbert-Kunz multiplicity is upper semi-continuous in S if and only if it is upper semi-continuous in R .*

Proof. Let Q be any prime in S and let $\mathfrak{p} = Q \cap R$. Note that $R_{\mathfrak{p}} \rightarrow S_Q$ is faithfully flat with regular fibers, so, by a result of Kunz ([10, Proposition 3.9]), $e_{HK}(R_{\mathfrak{p}}) = e_{HK}(S_Q)$. Thus, under our assumption, the Hilbert-Kunz multiplicity is constant in fibers.

Suppose upper semi-continuity holds in S . Let a be any real number and consider the closed set $V(I) = \{Q \mid Q \in \text{Spec } S, e_{HK}(Q) \geq a\}$. The argument above tells us that for any $Q \in V(I)$ any minimal prime $(Q \cap R)S$ is also in $V(I)$. Hence we get that $V(I) = V(JS)$ where $J = I \cap R$.

We claim that $V(J) = \{\mathfrak{p} \mid \mathfrak{p} \in \text{Spec } R, e_{HK}(\mathfrak{p}) \geq a\}$. Note that $J \in \mathfrak{p}$ if and only if $JS \subseteq Q$ for any prime Q in S that contracts to \mathfrak{p} , i.e. $e_{HK}(\mathfrak{p}) = e_{HK}(Q) \geq a$.

For the other direction, note that $f^*: \text{Spec } S \rightarrow \text{Spec } R$ is surjective, so, since e_{HK} is constant in fibers, we obtain that

$$\{Q \mid Q \in \text{Spec } S, e_{HK}(Q) < a\} = (f^*)^{-1}\{\mathfrak{p} \mid \mathfrak{p} \in \text{Spec } R, e_{HK}(\mathfrak{p}) < a\}.$$

Hence it is open. □

3. GLOBALLY UNIFORM HILBERT-KUNZ ESTIMATES FOR F-FINITE RINGS

In this section we essentially rebuild Tucker's uniform Hilbert-Kunz estimates from [15] in order to control the rate of convergence of the Hilbert-Kunz function on an open subset.

We will need the following facts about the Hilbert-Samuel multiplicity. See [8, Proposition 11.1.10, Theorem 11.2.4, Proposition 11.2.9] for proofs.

Proposition 13. *Let (R, \mathfrak{m}) be a local ring of dimension d , \underline{x} be a system of parameters and I an arbitrary \mathfrak{m} -primary ideal.*

- (1) $\ell(R/\underline{x}) \geq e(\underline{x}, R)$. If \underline{x} is a regular sequence, then equality holds.
- (2) (Associativity formula) $e(I, R) = \sum_{\mathfrak{p}} e(I, R/\mathfrak{p}) \ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}})$, where the sum is taken over all primes \mathfrak{p} , such that $\dim R/\mathfrak{p} = \dim R$.
- (3) For any numbers n_1, \dots, n_d , $e((x_1^{n_1}, \dots, x_d^{n_d}), R) = n_1 \dots n_d e(\underline{x}, R)$.

Lemma 14 (Key lemma). *Let R be an excellent ring of characteristic $p > 0$ and \mathfrak{p} a prime ideal of R . Let M be a finite R -module. There exists a constant C (depending only on M) and an element $s \notin \mathfrak{p}$, such that for any prime ideal $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$ and for all q , we have*

$$\ell_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}/\mathfrak{q}^{[q]}M_{\mathfrak{q}}) \leq Cq^{\dim M_{\mathfrak{q}}}.$$

Proof. Assume that $M = R/P$ is a cyclic module for a prime ideal P . If \mathfrak{p} does not contain P , we can invert any $s \in P \setminus \mathfrak{p}$, so $M_s = 0$ and the assertion is trivially true. Hence, assume $P \subseteq \mathfrak{p}$.

First, invert an element to make R/\mathfrak{p} regular; this is possible since R/\mathfrak{p} is an excellent domain. Let $S = R/P$, then $S/\mathfrak{p}S \cong R/\mathfrak{p}$ is regular too.

Consider the associated graded ring $\text{gr}_{\mathfrak{p}}(S) = \bigoplus_n \mathfrak{p}^n S / \mathfrak{p}^{n+1} S$. This is a finitely generated $S/\mathfrak{p}S$ -algebra, so by Generic Freeness ([11, 22.A]), we can invert an element of $S/\mathfrak{p}S$ and make it free over the regular ring $S/\mathfrak{p}S$. It follows that $\mathfrak{p}^n S / \mathfrak{p}^{n+1} S$ are projective $S/\mathfrak{p}S$ -modules for all n . Hence, by induction, using the sequences

$$0 \rightarrow \mathfrak{p}^n S / \mathfrak{p}^{n+1} S \rightarrow S / \mathfrak{p}^{n+1} S \rightarrow S / \mathfrak{p}^n S \rightarrow 0,$$

we get that all residue rings $S/\mathfrak{p}^n S$ are Cohen-Macaulay in this localization.

Let \mathfrak{q} be an arbitrary prime ideal in the obtained localization that contains \mathfrak{p} . Since $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ is a regular local ring, there exists a system of parameters \underline{x} that generates $\mathfrak{q}R_{\mathfrak{q}}$ modulo $\mathfrak{p}R_{\mathfrak{q}}$. Suppose \mathfrak{p} can be generated by t elements in R . Since $(\mathfrak{p}^{tq}, \underline{x}^{[q]})R_{\mathfrak{q}} \subseteq \mathfrak{q}^{[q]}R_{\mathfrak{q}}$,

$$\ell_{R_{\mathfrak{q}}}(S_{\mathfrak{q}}/\mathfrak{q}^{[q]}S_{\mathfrak{q}}) \leq \ell_{R_{\mathfrak{q}}}(S_{\mathfrak{q}}/(\mathfrak{p}^{tq}, (\underline{x})^{[q]})S_{\mathfrak{q}}).$$

Since S/\mathfrak{p}^{tq} are Cohen-Macaulay,

$$\ell_{R_{\mathfrak{q}}}\left(S_{\mathfrak{q}}/(\mathfrak{p}^{tq}, (\underline{x})^{[q]})S_{\mathfrak{q}}\right) = e((\underline{x})^{[q]}, S_{\mathfrak{q}}/\mathfrak{p}^{tq}S_{\mathfrak{q}}) = q^{\text{ht } \mathfrak{q}/\mathfrak{p}} e(\underline{x}, S_{\mathfrak{q}}/\mathfrak{p}^{tq}S_{\mathfrak{q}}).$$

Thus, using the associativity formula, we get

$$\ell_{R_{\mathfrak{q}}}\left(S_{\mathfrak{q}}/(\mathfrak{p}^{tq}, (\underline{x})^{[q]})S_{\mathfrak{q}}\right) = q^{\text{ht } \mathfrak{q}/\mathfrak{p}} e(\underline{x}, S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) \ell_{R_{\mathfrak{p}}}(S_{\mathfrak{p}}/\mathfrak{p}^{tq}S_{\mathfrak{p}}) = q^{\text{ht } \mathfrak{q}/\mathfrak{p}} \ell_{R_{\mathfrak{p}}}(S_{\mathfrak{p}}/\mathfrak{p}^{tq}S_{\mathfrak{p}}).$$

Note, that $e(\underline{x}, S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) = 1$, since \underline{x} generates the maximal ideal of a regular local ring $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$.

To finish the argument, note that $\ell_{R_{\mathfrak{p}}}(S_{\mathfrak{p}}/\mathfrak{p}^n S_{\mathfrak{p}}) = \ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/(P + \mathfrak{p}^n)R_{\mathfrak{p}})$ is a polynomial in n of degree $\text{ht } \mathfrak{p}/P$ for all sufficiently large n , so, clearly, there exists a constant D , such that for all q

$$\ell_{R_{\mathfrak{q}}}(S_{\mathfrak{p}}/\mathfrak{p}^{tq}S_{\mathfrak{p}}) = \ell_{R_{\mathfrak{q}}}(R_{\mathfrak{p}}/(P + \mathfrak{p}^{tq})R_{\mathfrak{p}}) \leq D(tq)^{\text{ht } \mathfrak{p}/P}.$$

Thus, we obtained a bound

$$\ell_{R_q}(S_q/\mathfrak{q}^{[q]}S_q) \leq \ell(S_q/(\mathfrak{p}^{tq}, (\underline{x})^{[q]})S_q) \leq q^{\text{ht } \mathfrak{q}/\mathfrak{p}} D(tq)^{\text{ht } \mathfrak{p}/P} = (Dt^{\text{ht } \mathfrak{p}/P}) q^{\text{ht } \mathfrak{q}/P} = Cq^{\dim S_q}.$$

Hence, the statement has been proved for $C = Dt^{\text{ht } \mathfrak{p}/P}$, a constant independent of \mathfrak{q} .

By choosing a prime filtration of M over R , we can reduce the general case to $M = R/P$. Namely, if P_i are prime ideals appearing in the prime filtration, then

$$\ell_{R_q}(M_q/\mathfrak{q}^{[q]}M_q) \leq \sum_i \ell_{R_q}((R/P_i)_q/\mathfrak{q}^{[q]}(R/P_i)_q).$$

Since there are finitely many primes P_i , we can invert finitely many elements in order to force the claim for all R/P_i . Also, note that $\dim M_q$ is the maximum of $\dim R_q/P_i R_q$ over the primes in a prime filtration. So,

$$\ell_{R_q}(M_q/\mathfrak{q}^{[q]}M_q) \leq \sum_i \ell_{R_q}(R_q/(P_i + \mathfrak{q}^{[q]})R_q) \leq \sum_i C_i q^{\text{ht } \mathfrak{q}/P_i} \leq \left(\sum_i C_i \right) q^{\dim M_q}.$$

□

Using a standard argument ([15, Lemma 3.3] or [12, Lemma 1.3]), we derive from the Key lemma the following result.

Corollary 15. *Let R be an excellent ring of characteristic $p > 0$ and \mathfrak{p} be a prime ideal of R . Suppose M and N are finite R -modules such that their localizations at every minimal prime are isomorphic. Then there exists a constant C and an element $s \notin \mathfrak{p}$, such that for any prime ideal $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$ and for all q , we have*

$$|\ell_{R_q}(M_q/\mathfrak{q}^{[q]}M_q) - \ell_{R_q}(N_q/\mathfrak{q}^{[q]}N_q)| \leq Cq^{\text{ht } \mathfrak{q}-1}.$$

Proof. By the assumptions, we have an exact sequence

$$N \rightarrow M \rightarrow K \rightarrow 0,$$

where $K_P = 0$ for every minimal prime P . By Lemma 14, we can find an element s_1 such that for some constant C_1 and all $\mathfrak{q} \in D_{s_1} \cap V(\mathfrak{p})$

$$\ell_{R_q}(M_q/\mathfrak{q}^{[q]}M_q) - \ell_{R_q}(N_q/\mathfrak{q}^{[q]}N_q) \leq \ell_{R_q}(K_q/\mathfrak{q}^{[q]}K_q) \leq C_1 q^{\dim K_q}.$$

Since $K_P = 0$ for any minimal prime P , $\dim K_q \leq \text{ht } \mathfrak{q} - 1$.

To finish the proof, we switch M and N in the first part of the argument, i.e. apply it to the sequence

$$M \rightarrow N \rightarrow L \rightarrow 0.$$

Hence, by inverting an element s_2 , we will get

$$\ell_{R_q}(N_q/\mathfrak{q}^{[q]}M_q) - \ell_{R_q}(M_q/\mathfrak{q}^{[q]}M_q) \leq \ell_{R_q}(L_q/\mathfrak{q}^{[q]}L_q) \leq C_2 q^{\dim L_q} \leq C_2 q^{\text{ht } \mathfrak{q}-1},$$

and the claim follows for $C = \max(C_1, C_2)$ and $s = s_1 s_2$. □

Definition 16. Let R be a ring of characteristic $p > 0$. For a prime ideal \mathfrak{p} of R , we denote $\alpha(\mathfrak{p}) = \log_p[k(\mathfrak{p}) : k(\mathfrak{p})^p]$, where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is the residue field of \mathfrak{p} .

We will need the following result of Kunz ([10, 2.3]).

Proposition 17. *Let R be F -finite and let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals. Then $\alpha(\mathfrak{p}) = \alpha(\mathfrak{q}) + \text{ht } \mathfrak{q}/\mathfrak{p}$.*

Theorem 18. *Let R be an F -finite domain and let \mathfrak{p} be an arbitrary prime ideal. Then there exists an element $s \notin \mathfrak{p}$ such that for any $\varepsilon > 0$ there is q_0 such that for all $q > q_0$*

$$\left| \ell_{R_q} (R_q / \mathfrak{q}^{[q]} R_q) / q^{\text{ht } \mathfrak{q}} - e_{HK}(\mathfrak{q}) \right| < \varepsilon$$

for all prime ideals $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$.

Proof. Since R is F -finite, $R^{p^{\alpha(0)}}$ and $R^{1/p}$ are isomorphic localized at the minimal prime 0. So, by Corollary 15, we can invert an element and obtain a global bound

$$\left| \ell_{R_q} \left(R_q^{p^{\alpha(0)}} / \mathfrak{q}^{[q]} R_q^{p^{\alpha(0)}} \right) - \ell_{R_q} \left(R_q^{1/p} / \mathfrak{q}^{[q]} R_q^{1/p} \right) \right| < C q^{\text{ht } \mathfrak{q}-1},$$

for an arbitrary prime ideal \mathfrak{q} containing \mathfrak{p} .

Now, to finish the proof, we follow Tucker's argument from [15]. Proposition 17 applied to the formula above gives

$$\begin{aligned} & \left| p^{\text{ht } \mathfrak{q} + \alpha(\mathfrak{q})} \ell_{R_q} (R_q / \mathfrak{q}^{[q]} R_q) - p^{\alpha(\mathfrak{q})} \ell_{R_q} (R_q / \mathfrak{q}^{[qp]} R_q) \right| < C q^{\text{ht } \mathfrak{q}-1}, \text{ so} \\ (1) \quad & \left| p^{\text{ht } \mathfrak{q}} \ell_{R_q} (R_q / \mathfrak{q}^{[q]} R_q) - \ell_{R_q} (R_q / \mathfrak{q}^{[qp]} R_q) \right| < p^{-\alpha(\mathfrak{q})} C q^{\text{ht } \mathfrak{q}-1} \leq C q^{\text{ht } \mathfrak{q}-1}. \end{aligned}$$

Now, we prove by induction on q' that

$$(2) \quad \left| (q')^{\text{ht } \mathfrak{q}} \ell_{R_q} (R_q / \mathfrak{q}^{[q]} R_q) - \ell_{R_q} (R_q / \mathfrak{q}^{[qq']} R_q) \right| < C (qq'/p)^{\text{ht } \mathfrak{q}-1} \frac{q' - 1}{p - 1}.$$

The induction base of $q' = p$ is (1). Now, assume that the claim holds for q' and we want to prove it for $q'p$.

First, (1) applied to qq' gives

$$(3) \quad \left| p^{\text{ht } \mathfrak{q}} \ell_{R_q} \left(R_q / \mathfrak{q}^{[qq']} R_q \right) - \ell_{R_q} \left(R_q / \mathfrak{q}^{[qq'p]} R_q \right) \right| < C (qq')^{\text{ht } \mathfrak{q}-1},$$

and, multiplying the induction hypothesis by $p^{\text{ht } \mathfrak{q}}$, we get

$$(4) \quad \left| (q'p)^{\text{ht } \mathfrak{q}} \ell_{R_q} \left(R_q / \mathfrak{q}^{[q]} R_q \right) - p^{\text{ht } \mathfrak{q}} \ell_{R_q} \left(R_q / \mathfrak{q}^{[qq']} R_q \right) \right| < C (qq')^{\text{ht } \mathfrak{q}-1} \frac{pq' - p}{p - 1}.$$

Combining (3) and (4) results in

$$\left| (q')^{\text{ht } \mathfrak{q}} \ell_{R_q} \left(R_q / \mathfrak{q}^{[q]} R_q \right) - \ell_{R_q} \left(R_q / \mathfrak{q}^{[qq']} R_q \right) \right| < C (qq')^{\text{ht } \mathfrak{q}-1} \left(\frac{q'p - p}{p - 1} + 1 \right),$$

and the induction step follows.

Now, dividing (2) by $q'^{\text{ht } \mathfrak{q}}$, we obtain

$$\left| \ell_{R_q} \left(R_q / \mathfrak{q}^{[q]} R_q \right) - \frac{1}{q'^{\text{ht } \mathfrak{q}}} \ell_{R_q} \left(R_q / \mathfrak{q}^{[qq']} R_q \right) \right| < C q^{\text{ht } \mathfrak{q}-1} \cdot \frac{q' - 1}{p - 1} \cdot \frac{1}{q' p^{\text{ht } \mathfrak{q}-1}} \leq C q^{\text{ht } \mathfrak{q}-1}.$$

Thus, if we let $q' \rightarrow \infty$, we get that

$$\left| \ell_{R_q} (R_q / \mathfrak{q}^{[q]} R_q) - q^{\text{ht } \mathfrak{q}} e_{HK}(\mathfrak{q}) \right| < C q^{\text{ht } \mathfrak{q}-1},$$

and the claim follows. \square

4. UNIFORM ESTIMATES FOR A FLAT EXTENSION

In this section we prove convergence estimates of Theorem 18 for algebras of essentially finite type over a complete domain. To do so, we use existence of a faithfully flat F-finite extension, and we relativize the estimates of the previous section to use in the extension.

Lemma 19. *Let R be a locally equidimensional excellent ring and S be an R -algebra. Let I be an ideal in R , let M be an S -module such that $\text{Supp } M \subseteq V(I)$, and \mathfrak{p} be a prime ideal of R . Then there exists an element $s \notin \mathfrak{p}$ and a constant C such that for any prime ideal $\mathfrak{q} \in V(\mathfrak{p}) \cap D(s)$ and for any prime ideal Q in S minimal over $\mathfrak{q}S$*

$$\ell_{S_Q}(M_Q/\mathfrak{q}^{[q]}M_Q) \leq Cq^{\text{ht } \mathfrak{q} - \text{ht } I} \ell_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

Proof. If I is not contained in \mathfrak{p} we can invert an element and make M to be zero. So assume $I \subseteq \mathfrak{p}$.

Since R is excellent, we can invert an element $s \notin \mathfrak{p}$ to make R/\mathfrak{p} regular and $R/(\mathfrak{p}^n + I)$ to be Cohen-Macaulay for all n (see the proof of Lemma 14). We claim that the required bound holds for this s .

By taking a prime filtration of M we reduce the statement to $M = S/J$, where J is a prime ideal in S that contains IS . So

$$\ell_{S_Q}(S_Q/(\mathfrak{q}^{[q]}S + J)S_Q) \leq \ell_{S_Q}(S_Q/(\mathfrak{q}^{[q]} + I)S_Q).$$

By tensoring a prime filtration of $R_{\mathfrak{q}}/(\mathfrak{q}^{[q]} + I)R_{\mathfrak{q}}$ with S_Q , we have

$$\ell_{S_Q}(S_Q/(\mathfrak{q}^{[q]} + I)S_Q) \leq \ell_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/(\mathfrak{q}^{[q]} + I)R_{\mathfrak{q}}) \ell_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

Since R/\mathfrak{p} is regular, we can write $\mathfrak{q}R_{\mathfrak{q}} = (\mathfrak{p} + (\underline{x}))R_{\mathfrak{q}}$, where \underline{x} are minimal generators of $\mathfrak{q}/\mathfrak{p}$. Suppose \mathfrak{p} can be generated by t elements in R , hence $\mathfrak{p}^{tq} \subseteq \mathfrak{p}^{[q]}$. Thus

$$\ell_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/(\mathfrak{q}^{[q]} + I)R_{\mathfrak{q}}) = \ell_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/(\mathfrak{p}^{[q]} + (\underline{x})^{[q]} + I)R_{\mathfrak{q}}) \leq \ell_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/(\mathfrak{p}^{tq} + (\underline{x})^{[q]} + I)R_{\mathfrak{q}}).$$

Now, since $R/\mathfrak{p}^{tq} + I$ are Cohen-Macaulay,

$$\ell_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/(\mathfrak{p}^{tq} + I + (\underline{x})^{[q]})R_{\mathfrak{q}}) = e((\underline{x})^{[q]}, R_{\mathfrak{q}}/(\mathfrak{p}^{tq} + I)R_{\mathfrak{q}}).$$

Moreover, by the associativity formula,

$$e((\underline{x})^{[q]}, R_{\mathfrak{q}}/(\mathfrak{p}^{tq} + I)R_{\mathfrak{q}}) = e((\underline{x})^{[q]}, R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) \ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/(\mathfrak{p}^{tq} + I)R_{\mathfrak{p}}),$$

and, using that R/\mathfrak{p} is regular,

$$e((\underline{x})^{[q]}, R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) \ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/(\mathfrak{p}^{tq} + I)R_{\mathfrak{p}}) = q^{\text{ht } \mathfrak{q}/\mathfrak{p}} \ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/(\mathfrak{p}^{tq} + I)R_{\mathfrak{p}}).$$

The Hilbert-Samuel polynomial of $R_{\mathfrak{p}}/IR_{\mathfrak{p}}$ has degree $\dim R_{\mathfrak{p}}/I = \text{ht } \mathfrak{p} - \text{ht } I$. Hence we can find a constant D such that

$$\ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/(\mathfrak{p}^{tq} + I)R_{\mathfrak{p}}) \leq D(tq)^{\text{ht } \mathfrak{p} - \text{ht } I} = Cq^{\text{ht } \mathfrak{p} - \text{ht } I}$$

and the claim follows. \square

We will need the following lemma about the Gamma construction. It is a step in the proof of [7, Lemma 6.13], and a more detailed exposition can be found in Hochster's notes ([6, Theorem, page 139]).

Lemma 20. *Let B be a complete local domain and S be a B -algebra of essentially finite type. Suppose S is a domain then there exists a purely inseparable faithfully flat F -finite B -algebra B^Γ such that $S \otimes_B B^\Gamma$ is a domain.*

Theorem 21. *Let B be a complete local domain and R be a domain and a B -algebra of essentially finite type. Let \mathfrak{p} be an arbitrary prime ideal in R , then there exists an element $s \notin \mathfrak{p}$ such that for any $\varepsilon > 0$ there is q_0 such that for all $q > q_0$*

$$|\ell_{R_q}(R_q/\mathfrak{q}^{[q]}R_q)/q^{\text{ht } \mathfrak{q}} - e_{HK}(\mathfrak{q})| < \varepsilon$$

for all prime ideals $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$.

Proof. We apply Lemma 20 to the quotient field L of R and obtain a B -algebra B^Γ . Note that $S = R \otimes_B B^\Gamma$ is F -finite, so $S^{1/p}$ is a finitely generated S -module.

By Lemma 20, $S \otimes_R L \cong B^\Gamma \otimes_B R \otimes_R L \cong B^\Gamma \otimes_B L$ is a domain. Since B^Γ is purely inseparable over B , $B^\Gamma \otimes_B L$ is integral over a field L , so it is a field. Since taking p -roots commutes with localization, $(S)^{1/p} \otimes_R L \cong (S \otimes_R L)^{1/p}$, so it is a free module over the field $S \otimes_R L \cong B^\Gamma \otimes_B L$. Hence, we can invert an element f of R to make $S_f^{1/p}$ a free module over S_f . Since R is a subring of S and $S \otimes_R L$ is a field, $S \otimes_R L$ is the quotient field of S , thus, by definition, the rank of the free module $(S \otimes_R L)^{1/p}$ is $p^{\alpha(0)}$.

Therefore there exist maps

$$0 \rightarrow S^{1/p} \rightarrow S^{p^{\alpha(0)}} \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow S^{p^{\alpha(0)}} \rightarrow S^{1/p} \rightarrow N \rightarrow 0$$

such that $\text{Supp } M, \text{Supp } N \subseteq V(fS)$.

Thus, using Lemma 19 to M and N , we can invert an element s and obtain that, for any prime \mathfrak{q} containing \mathfrak{p} and for any minimal prime Q of $\mathfrak{q}S$,

$$\begin{aligned} \ell_{S_Q}(S_Q^{p^{\alpha(0)}}/\mathfrak{q}^{[q]}S_Q^{p^{\alpha(0)}}) - \ell_{S_Q}(S_Q^{1/p}/\mathfrak{q}^{[q]}S_Q^{1/p}) &\leq \ell_{S_Q}(M_Q/\mathfrak{q}^{[q]}M_Q) \leq C_1 q^{\text{ht } \mathfrak{q} - \text{ht}(f)} \ell_{S_Q}(S_Q/\mathfrak{q}S_Q), \\ \ell_{S_Q}(S_Q^{1/p}/\mathfrak{q}^{[q]}S_Q^{1/p}) - \ell_{S_Q}(S_Q^{p^{\alpha(0)}}/\mathfrak{q}^{[q]}S_Q^{p^{\alpha(0)}}) &\leq \ell_{S_Q}(N_Q/\mathfrak{q}^{[q]}N_Q) \leq C_2 q^{\text{ht } \mathfrak{q} - \text{ht}(f)} \ell_{S_Q}(S_Q/\mathfrak{q}S_Q). \end{aligned}$$

Thus, by taking $C = \max(C_1, C_2)$ and noting that $\text{ht}(f) = 1$, we obtain

$$\left| \ell_{S_Q}(S_Q^{p^{\alpha(0)}}/\mathfrak{q}^{[q]}S_Q^{p^{\alpha(0)}}) - \ell_{S_Q}(S_Q^{1/p}/\mathfrak{q}^{[q]}S_Q^{1/p}) \right| < C q^{\text{ht } \mathfrak{q} - 1} \ell_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

So, since $\alpha(0) = \text{ht } Q + \alpha(Q)$ by Proposition 17,

$$|p^{\text{ht } Q + \alpha(Q)} \ell_{S_Q}(S_Q/\mathfrak{q}^{[q]}S_Q) - p^{\alpha(Q)} \ell_{S_Q}(S_Q/\mathfrak{q}^{[qp]}S_Q)| < C q^{\text{ht } \mathfrak{q} - 1} \ell_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

Note that S_Q is flat over R_q and $\mathfrak{q}S_Q$ is Q -primary. Hence for any artinian R_q -module M ,

$$\ell_{S_Q}(M \otimes_{R_q} S_Q) = \ell_{R_q}(M) \ell_{S_Q}(S_Q/\mathfrak{q}S_Q).$$

Therefore, the estimate above can be rewritten as

$$|p^{\text{ht } Q + \alpha(Q)} \ell_{R_q}(R_q/\mathfrak{q}^{[q]}R_q) - p^{\alpha(Q)} \ell_{R_q}(R_q/\mathfrak{q}^{[qp]}R_q)| < C q^{\text{ht } \mathfrak{q} - 1}.$$

Since S is flat $\text{ht } Q = \text{ht } \mathfrak{q}$, so we obtain Equation 1 from Theorem 18:

$$|p^{\text{ht } \mathfrak{q}} \ell_{R_q}(R_q/\mathfrak{q}^{[q]}R_q) - \ell_{R_q}(R_q/\mathfrak{q}^{[qp]}R_q)| < C p^{-\alpha(Q)} q^{\text{ht } \mathfrak{q} - 1} \leq C q^{\text{ht } \mathfrak{q} - 1};$$

and the proof follows the argument in Theorem 18. \square

5. PROOF OF THE MAIN RESULT AND CONCLUDING REMARKS

Now, we want to finish the proof of upper semi-continuity of the Hilbert-Kunz multiplicity for F -finite rings and algebras of essentially finite type over an excellent local ring. To do this, we verify the second statement of Proposition 9.

Theorem 22. *Let R be a locally equidimensional ring. Suppose that R is either F -finite or is an algebra of essentially finite type over an excellent local ring B . If \mathfrak{p} be a prime ideal of R , then for any $\varepsilon > 0$, there exists $s \notin \mathfrak{p}$, such that for all prime ideals $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$*

$$e_{HK}(\mathfrak{q}) < e_{HK}(\mathfrak{p}) + \varepsilon.$$

Proof. If R is not F -finite, first, consider extension $R \rightarrow R \otimes_B \widehat{B}$. Since B is excellent, the natural map $B \rightarrow \widehat{B}$ is regular. So, by [11, Lemma 4, p. 253], $R \rightarrow R \otimes_B \widehat{B}$ satisfies the conditions of Lemma 12. Hence, by Proposition 9 and Lemma 12, we assume that B is complete.

Note that the classes of rings that we consider are stable under taking quotients. So, by Proposition 10, we can assume that R is a domain.

By Theorem 18 and Theorem 21, there exists an element $s \notin \mathfrak{p}$ and a fixed power $q_0 = p^e$, such that for all $\mathfrak{q} \in D_s \cap V(\mathfrak{p})$

$$\left| \ell_{R_{\mathfrak{q}}} (R_{\mathfrak{q}}/\mathfrak{q}^{[q_0]} R_{\mathfrak{q}}) / q_0^{\text{ht } \mathfrak{q}} - e_{HK}(\mathfrak{q}) \right| < \varepsilon/2.$$

In particular,

$$\left| \ell_{R_{\mathfrak{p}}} (R_{\mathfrak{p}}/\mathfrak{p}^{[q_0]} R_{\mathfrak{p}}) / q_0^{\text{ht } \mathfrak{p}} - e_{HK}(\mathfrak{p}) \right| < \varepsilon/2.$$

Now, we can use Proposition 6, and obtain a non-empty subset $\mathfrak{p} \in U \subseteq V(\mathfrak{p})$ open in $V(\mathfrak{p})$ such that for any $\mathfrak{q} \in U$,

$$\ell_{R_{\mathfrak{q}}} (R_{\mathfrak{q}}/\mathfrak{q}^{[q_0]} R_{\mathfrak{q}}) / q_0^{\text{ht } \mathfrak{q}} = f_{q_0}(\mathfrak{q}) = f_{q_0}(\mathfrak{p}) = \ell_{R_{\mathfrak{p}}} (R_{\mathfrak{p}}/\mathfrak{p}^{[q_0]} R_{\mathfrak{p}}) / q_0^{\text{ht } \mathfrak{p}}.$$

For completeness we are giving a construction of such U below.

Since $R/\mathfrak{p}^{[q_0]}$ is excellent, its Cohen-Macaulay locus is open ([5, 7.8.3(iv)]). Thus we can find an open subset $U \subseteq V(\mathfrak{p})$ containing \mathfrak{p} such that for any $\mathfrak{q} \in U$, $(R/\mathfrak{p}^{[q_0]})_{\mathfrak{q}}$ is Cohen-Macaulay and $(R/\mathfrak{p})_{\mathfrak{q}}$ is regular.

Let \mathfrak{q} be an arbitrary prime in U . Since $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ is regular, $\mathfrak{q}R_{\mathfrak{q}}$ is generated by a regular sequence \underline{x} modulo $\mathfrak{p}R_{\mathfrak{q}}$. Then, by the associativity formula,

$$\ell_{R_{\mathfrak{q}}} (R_{\mathfrak{q}}/\mathfrak{q}^{[q_0]} R_{\mathfrak{q}}) = \ell_{R_{\mathfrak{q}}} (R_{\mathfrak{q}}/(\mathfrak{p}^{[q_0]}, \underline{x}^{[q_0]}) R_{\mathfrak{q}}) = e((\underline{x})^{[q_0]}, R_{\mathfrak{q}}/\mathfrak{p}^{[q_0]}) = q_0^{\text{ht } \mathfrak{q}/\mathfrak{p}} \ell_{R_{\mathfrak{p}}} (R_{\mathfrak{p}}/\mathfrak{p}^{[q_0]} R_{\mathfrak{p}}).$$

Thus, we obtain that on $U \cap D_s$, $\ell_{R_{\mathfrak{p}}} (R_{\mathfrak{p}}/\mathfrak{p}^{[q_0]} R_{\mathfrak{p}}) / q_0^{\text{ht } \mathfrak{p}}$ is within $\varepsilon/2$ from both $e_{HK}(\mathfrak{p})$ and $e_{HK}(\mathfrak{q})$ and the statement follows. \square

Corollary 23. *Let R be a locally equidimensional ring. Moreover, suppose that either R is F -finite or is an algebra of essentially finite type over an excellent local ring B . Then the Hilbert-Kunz multiplicity is upper semi-continuous on $\text{Spec } R$.*

We note the following corollary of semi-continuity.

Corollary 24. *Let R be a Noetherian ring and suppose the Hilbert-Kunz multiplicity is upper semi-continuous on $\operatorname{Spec} R$. Then the Hilbert-Kunz multiplicity satisfies the ascending chain condition on $\operatorname{Spec} R$, i.e. any increasing sequence $e_1 = e_{HK}(\mathfrak{p}_1) \leq e_2 = e_{HK}(\mathfrak{p}_2) \leq \dots$ stabilizes.*

In particular, the Hilbert-Kunz multiplicity attains its maximum on $\operatorname{Spec} R$.

Proof. Since e_{HK} is upper semi-continuous $U_i = \{\mathfrak{p} \mid e_{HK}(\mathfrak{p}) < e_i\}$ form an increasing sequence of open sets, so it stabilizes. \square

Remark 25. In [15], Kevin Tucker asked if F-signature is lower semi-continuous in F-finite rings. One could hope that the ideas of this paper are extendable for F-signature, but, at the present moment, we know nothing about the convergence rate of the F-signature of a local ring.

In fact, one could even ask if the splitting numbers can be written as

$$a_e = r_F q^h + O(q^{h-1}),$$

where r_F is the F -splitting ratio, $h = \alpha(R) + \dim(R/P)$, and P is the splitting prime of R , see [15] for more details.

Remark 26. We proved Conjecture 4 for the F-finite case and algebras over an excellent local ring and want to discuss further difficulties. At the present moment, the author does not see any way to prove the conjecture in full generality, for an arbitrary excellent ring.

The problem stems from the known proof of existence of the Hilbert-Kunz multiplicity, both the original paper ([12]) and its refinement ([15]) prove existence of the limit for a local ring by reducing to a faithfully flat F-finite extension obtained by extending the residue field. Thus, there is not much connection between these objects for different localizations, so the results and methods of the present paper cannot be applied.

Furthermore, it is not enough to have a global faithfully flat F-finite extension; we needed to use the Gamma construction in order to have an extension that has suitable properties.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904-4137 USA
E-mail address: is6eu@virginia.edu